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Path integrals for potential problems with δ-function perturbation

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Abstract. In this paper I present several examples of potential problems with a δ-function perturbation by means of path integrals. The idea is to sum a perturbation series expansion resulting in an energy-dependent Green function $G(E)$. The energy levels E_n of the perturbed problem are determined by the equation (one-dimensional case) $\mathrm{i}G^{(V)}(a, a; E_n) = \hbar/\gamma$, where $G^{(V)}$ is the Green function of the unperturbed problem, γ is the strength of the δ potential and a it's location in $\mathbb{R}$. In D-dimensional radial problems with a spherically shaped δ function located at $r = a$ this equation changes into $\mathrm{i}G_l^{(V)}(a, a; E_n) = \hbar/a^{D-1}\gamma$, where l denotes the angular momentum number.

1. Introduction

In recent years there has been an enormous achievement in solving Feynman path integrals exactly. Besides the harmonic oscillator, whose solution is originally due to Feynman himself [1] (or, the general quadratic Lagrangian, e.g. [2]), the radial harmonic oscillator [3] and problems related to homogeneous spaces (rotator [4], pseudosphere [5] and (modified) Pöschl-Teller potential [6, 7]), a large number of problems could be treated by the celebrated spacetime transformation technique introduced by Duru and Kleinert [8] in their treatment of the hydrogen atom. From these basic solutions a path integral problem is exactly solvable if the corresponding Schrödinger equation is equivalent to the differential equation for the confluent hypergeometric function or the hypergeometric function.

However, there are some potential and boundary problems which do not fall into these two classes. One such problem is the δ-function potential and potential problems with a δ-function perturbation. As it turns out, most of these problems are not even solvable in the sense that wavefunctions and energy levels can be explicitly stated. But it is possible to derive an exact (and in general) transcendental equation which determines wavefunctions and energy levels in a unique way.

This kind of problem turns out to be useful in the study of collision theory, where ionization is also considered, e.g. [9-12].

In this paper I want to present how these transcendental equations can be derived by means of path integral technique and perturbation series summation. The scope is thus to give a systematic and consistent approach to this kind of problem in the framework of path integrals.

The paper is organized as follows. In section 2, I present the general method. The main step in the path integral calculation involves an expansion in a perturbation series which turns out to be exactly summable. Because I am dealing with arbitrary potentials with a δ-function perturbation this is clearly a generalization of [13, 14], where only the simplest case (δ function in $\mathbb{R}$) was treated.

In section 3 several applications are discussed which include:

(i) One-dimensional examples: (a) free particle; (b) free particle on the half-line; (c) infinite square well; (d) harmonic oscillator; (e) harmonic oscillator on the half-line; (f) Morse potential and Liouville quantum mechanics; (g) reflectionless potential; (h) reflectionless potential on the half-line; (i) inverse-distance potential; (j) Pöschl–Teller potential; (k) modified Pöschl–Teller potential; (l) free particle with δ function;

(ii) D-dimensional radial examples: (a) free particle; (b) two-dimensional particle confined in a sector; (c) radial harmonic oscillator; (d) inverse distance ('Coulomb') potential.

In all the examples the (energy-dependent) Green function of the unperturbed problem is explicitly known and can be calculated by the path integral technique. Section 4 contains some concluding remarks and in the appendix the Feynman kernel for the perturbed reflectionless potential is explicitly calculated.

2. Summation of the perturbation expansion

2.1. One-dimensional problems

We consider an arbitrary potential $V(x)$ in one dimension with an additional δ-function perturbation such that

$$W(x) = V(x) - \gamma\delta(x-a). \tag{1}$$

The path integral for this potential problem has the form

$$\begin{aligned} K(x'', x'; \tau) &= \int \mathrm{D}x(t) \exp\left[-\frac{1}{\hbar}\int_0^\tau \left(\frac{m}{2}\dot{x}^2 + W(x)\right) \mathrm{d}t\right] \\ &= \lim_{N\to\infty} \left(\frac{m}{2\pi\varepsilon\hbar}\right)^{N/2} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} \mathrm{d}x^{(j)} \\ &\quad \times \exp\left[-\frac{1}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\varepsilon}\Delta^2 x^{(j)} + W(x^{(j)})\right)\right] \end{aligned} \tag{2}$$

where $x^{(j)} = x(j\varepsilon)$, $\varepsilon = \tau/N$, $\Delta x^{(j)} = x^{(j)} - x^{(j-1)}$ in the limit $N\to\infty$ and we have chosen 'imaginary' time. Let us assume that the path integral (Feynman kernel, respectively) for the potential V is known, i.e.

$$K^{(V)}(x'', x'; \tau) = \int \mathrm{D}x(t) \exp\left[-\frac{1}{\hbar}\int_0^\tau \left(\frac{m}{2}\dot{x}^2 + V(x)\right) \mathrm{d}t\right] \tag{3}$$

including, of course, the (energy-dependent) Green function

$$\begin{aligned} G^{(V)}(x'', x'; s) &= \int_0^\infty \exp(-s\tau/\hbar) K^{(V)}(x'', x'; \tau)\, \mathrm{d}\tau \\ K^{(V)}(x'', x'; \tau) &= \frac{1}{2\pi \mathrm{i}\hbar}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} G^{(V)}(x'', x'; s) \exp(s\tau/\hbar)\, \mathrm{d}s. \end{aligned} \tag{4}$$

Expanding (2) into a perturbation expansion according to [13, 14] yields

$$K(x'', x'; \tau) = K^{(V)}(x'', x'; \tau) + \sum_{n=1}^{\infty} \left(\frac{\gamma}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}x^{(j)} \int_0^{\tau} \mathrm{d}t^{(j)} \delta(x^{(j)} - a)$$
$$\times K^{(V)}(x^{(1)}, x'; t^{(1)}) \ldots K^{(V)}(x^{(n)}, x^{(n-1)}; t^{(n)} - t^{(n-1)})$$
$$\times K^{(V)}(x'', x^{(n)}; \tau - t^{(n)})$$
$$= K^{(V)}(x'', x'; \tau) + \sum_{n=1}^{\infty} \left(\frac{\gamma}{\hbar}\right)^n \int_0^{\tau} \mathrm{d}t^{(n)} \int_0^{t^{(n)}} \mathrm{d}t^{(n-1)} \ldots \int_0^{t^{(2)}} \mathrm{d}t^{(1)}$$
$$\times K^{(V)}(a, x'; t^{(1)}) \ldots K^{(V)}(a, a; t^{(n)} - t^{(n-1)}) K^{(V)}(x'', a; \tau - t^{(n)}). \tag{5}$$

In the second step I have ordered the time as $t^{(1)} < t^{(2)} < \ldots < \tau$ and paid attention to the fact that $K(t^{(j)} - t^{(j-1)})$ is different from zero only if $t^{(j)} > t^{(j-1)}$. Introducing the Green function $G(s)$ of the perturbed system similarly to (4), I obtain, due to the convolution theorem of Laplace transformation,

$$G(x'', x'; s) = G^{(V)}(x'', x'; s) + \sum_{n=1}^{\infty} \left(\frac{\gamma}{\hbar}\right)^n G^{(V)}(x'', a; s) G^{(V)}(a, x'; s) [G^{(V)}(a, a; s)]^{n-1}$$
$$= G^{(V)}(x'', x'; s) + \frac{\gamma}{\hbar} \frac{G^{(V)}(x'', a; s) G^{(V)}(a, x'; s)}{1 - (\gamma/\hbar) G^{(V)}(a, a; s)} \tag{6}$$

where it is assumed that $G^{(V)}(a, a; s)$ actually exists and the summation makes sense. The energy levels s_n of the perturbed problem $W(x)$ are therefore determined in a unique way by the equation

$$G^{(V)}(a, a; s_n) = \hbar/\gamma. \tag{7}$$

The same result can be achieved by solving the Schrödinger equation for the perturbed problem

$$-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \Psi(x) + V(x)\Psi(x) - E\Psi(x) = \gamma\delta(x - a)\Psi(x).$$

Translating (6) into 'real' time (perform the replacement: $\mathrm{i}T = \tau$ and $\sqrt{s} = -\mathrm{i}\sqrt{E}$) yields

$$G(x'', x'; E) = G^{(V)}(x'', x'; E) + \mathrm{i} \frac{\gamma}{\hbar} \frac{G^{(V)}(x'', a; E) G^{(V)}(a, x'; E)}{1 - \mathrm{i}(\gamma/\hbar) G^{(V)}(a, a; E)} \tag{8}$$

and the energy levels E_n are determined by

$$\hbar/\gamma = \mathrm{i} G^{(V)}(a, a; E_n). \tag{9}$$

Here $G(E)$ denotes the Green function defined by

$$G(x'', x'; E) = \int_0^{\infty} \exp(\mathrm{i}ET/\hbar) K(x'', x'; T)\, \mathrm{d}T. \tag{10}$$

Denoting by $E_n^{(V)}$ and $\Psi_n^{(V)}$ the energy levels and wavefunctions of the unperturbed problem, i.e.

$$G^{(V)}(x'', x'; E) = \sum_n \frac{\Psi_n^{(V)*}(x') \Psi_n^{(V)}(x'')}{E_n^{(V)} - E} \tag{11}$$

we find for the Green function of the perturbed problem

$$\operatorname{Res} G(x'', x'; E)\big|_{E = E_n^{(V)}} = 0. \tag{12}$$

Our result is in accordance with the operator approach of Bychkov [15] and Gaveau and Schulman [16]. Whereas in [15] a specific problem was discussed (periodic lattice of δ potentials), Gaveau and Schulman obtained for a potential like in (1) for the (time-dependent) Feynman kernel the implicit equation

$$K(x'', x'; T) = K^{(V)}(x'', x'; T) + \mathrm{i}\frac{\gamma}{\hbar}\int_0^T K^{(V)}(x'', a; t)K(a, x'; T-t)\,\mathrm{d}t. \tag{13}$$

Fourier transformation yields

$$G(x'', x'; E) = G^{(V)}(x'', x'; E) + \mathrm{i}\frac{\gamma}{\hbar}G^{(V)}(x'', a; E)G(a, x'; E) \tag{14}$$

and (6), (8) are recovered.

Taking in (6) the residuum at $s = s_i$ ($\equiv$ energy of the ith bound state) we obtain for the corresponding wavefunction

$$\Psi_i(x) = \left(\lim_{s\to s_i}\frac{s - s_i}{(\hbar/\gamma) - G^{(V)}(a, a; s)}\right)^{1/2} G(x, a; s_i). \tag{15}$$

2.2. *D-dimensional radial problems*

I consider in D dimensions a radial potential according to

$$W(r) = V(r) - \gamma\delta(r - a) \tag{16}$$

i.e. a spherically shaped δ-function perturbation of the potential $V(r)$. We must require $a \neq 0$ because point perturbation leads to the evaluation of Green functions, where both arguments are equal (and zero), an expression which in general does not exist (even for non-zero arguments). Of course, I am using the usual D-dimensional polar coordinates [17, chapter XI]:

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{D-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{D-2}\cos\phi \\
x_D &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{D-2}\sin\phi
\end{aligned} \tag{17}$$

where $0 \leqslant \theta_\nu \leqslant \pi$ ($\nu = 1, \ldots, D-2$), $0 \leqslant \phi \leqslant 2\pi$, $r \geqslant 0$. For a radial problem we can separate variables in the path integral:

$$\begin{aligned}
K(x'', x'; \tau) &= \int \mathrm{D}x(t)\exp\left[-\frac{1}{\hbar}\int_0^\tau\left(\frac{m}{2}\dot{x}^2 + W(r)\right)\mathrm{d}t\right] \\
&= \sum_{l=0}^{\infty} K_l(r'', r'; \tau)S_l^\mu(\Omega'')S_l^{\mu*}(\Omega')
\end{aligned} \tag{18}$$

where $S_l^\mu(\Omega)$ denote the real hyper-spherical harmonics of degree l with unit vector Ω and $l \in N_0$, $\mu = 1, \ldots, M$, $M = (2l+D-2)(l+D-3)!/l!(D-2)!$ and the radial path integral $K_l(\tau)$ is given by [3, 4, 18]

$$\begin{aligned} K_l(r'', r'; \tau) &= (r'r'')^{(1-D)/2} \int Dr(t)\mu_l^{(D)}[r^2] \exp\left[-\frac{1}{\hbar}\int_0^\tau \left(\frac{m}{2}\dot{r}^2 + W(r)\right) dt\right] \\ &= (r'r'')^{(1-D)/2} \lim_{N\to\infty} \left(\frac{m}{2\pi\varepsilon\hbar}\right)^{N/2} \prod_{j=1}^{N-1} \int_0^\infty dr^{(j)} \prod_{j=1}^{N} \mu_l^{(D)}[r^{(j-1)}r^{(j)}] \\ &\quad \times \exp\left[-\frac{1}{\hbar}\left(\frac{m}{2\varepsilon}\Delta^2 r^{(j)} + W(r^{(j)})\right)\right] \end{aligned} \tag{19}$$

with the functional measure

$$\begin{aligned} \mu_l^{(D)}[r^2] &= \lim_{N\to\infty} \mu_l^{(D)}[r^{(j-1)}r^{(j)}] \\ &= \lim_{N\to\infty} \prod_{j=1}^{N} \left(\frac{2\pi m r^{(j-1)} r^{(j)}}{\varepsilon\hbar}\right)^{1/2} \exp\left(-\frac{m r^{(j-1)} r^{(j)}}{\varepsilon\hbar}\right) I_{l+(D-2)/2}\left(\frac{m r^{(j-1)} r^{(j)}}{\varepsilon\hbar}\right). \end{aligned} \tag{20}$$

Proceeding in a similar way to the previous section and expanding the perturbed problem into a perturbation series yields

$$\begin{aligned} K(x'', x'; \tau) &= K^{(V)}(x'', x'; \tau) + \sum_{n=1}^\infty \left(\frac{\gamma}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^n \int_0^\infty [r^{(j)}]^{D-1} dr^{(j)} \delta(r^{(j)} - a) \\ &\quad \times \int d\Omega^{(j)} \int_0^\tau dt^{(j)} \\ &\quad \times \sum_{l^{(0)}=0}^\infty K_{l^{(0)}}^{(V)}(r^{(1)}, r'; t^{(1)}) S_{l^{(0)}}^{\mu^{(0)}}(\Omega^{(1)}) S_{l^{(0)}}^{\mu^{(0)}*}(\Omega') \\ &\quad \vdots \\ &\quad \times \sum_{l^{(n-1)}=0}^\infty K_{l^{(n-1)}}^{(V)}(r^{(n)}, r^{(n-1)}; t^{(n)} - t^{(n-1)}) S_{l^{(n-1)}}^{\mu^{(n-1)}}(\Omega^{(n)}) S_{l^{(n-1)}}^{\mu^{(n-1)}*}(\Omega^{(n-1)}) \\ &\quad \times \sum_{l^{(n)}=0}^\infty K_{l^{(n)}}^{(V)}(r'', r^{(n)}; \tau - t^{(n)}) S_{l^{(n)}}^{\mu^{(n)}}(\Omega'') S_{l^{(n)}}^{\mu^{(n)}*}(\Omega^{(n)}) \\ &= \sum_{l=0}^\infty K_l(r'', r'; \tau) S_l^\mu(\Omega'') S_l^{\mu*}(\Omega') \end{aligned} \tag{21}$$

with the radial perturbation expansion

$$\begin{aligned} K_l(r'', r', \tau) &= K_l^{(V)}(r'', r'; \tau) + \sum_{n=1}^\infty \left(\frac{a^{D-1}\gamma}{\hbar}\right)^n \int_0^\tau dt^{(n)} \ldots \int_0^{t^{(2)}} dt^{(1)} \\ &\quad \times K_l^{(V)}(a, r'; t^{(1)}) \ldots K_l^{(V)}(a, a; t^{(n)} - t^{(n-1)}) K_l^{(V)}(r'', a; \tau - t^{(n)}). \end{aligned} \tag{22}$$

Introducing the Green function $G(s)$ similar to (4), I obtain

$$G(x'', x'; s) = \sum_{l=0}^\infty G_l(r'', r'; \tau) S_l^\mu(\Omega'') S_l^{\mu*}(\Omega') \tag{23}$$

where the radial Green function is given by

$$G_l(r'', r'; s) = G_l^{(V)}(r'', r'; s) + \frac{G_l^{(V)}(r'', a; s) G_l^{(V)}(a, r'; s)}{\hbar / a^{D-1}\gamma - G_l^{(V)}(a, a; s)}. \tag{24}$$

Therefore the energy levels s_n are determined by the equation

$$\frac{\hbar}{a^{D-1}\gamma} = G_l^{(V)}(a, a; s_n). \tag{25}$$

Of course, this is in general an implicit (transcendental) equation. In 'real' time (25) has the form

$$\frac{\hbar}{a^{D-1}\gamma} = \mathrm{i} G_l^{(V)}(a, a; E_n). \tag{26}$$

The same result can be achieved by solving the radial Schrödinger equation for the perturbed problem

$$-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{D-1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\Psi(r) + \left(V(r) + \frac{\hbar^2 l(l+D-2)}{2mr^2} - E\right)\Psi(r) = \gamma\delta(r-a)\Psi(r).$$

The corresponding radial wavefunctions are similar to (15), given by

$$\Psi_{l,t}(r) = \left(\lim_{s\to s_t} \frac{s - s_t}{(\hbar/\gamma) a^{1-D} - G_l^{(V)}(a, a; s)}\right)^{1/2} G_l(r, a; s). \tag{27}$$

3. Applications

3.1. One-dimensional problems

3.1.1. Free particle. This easy example of the free particle (FP), where $V(x) \equiv 0$, has first been discussed by Goovaerts *et al* [19] (however, with a different perturbation expansion) and later on by Bauch [13] and Lawande and Bhagwat [14]. An operator treatment is due to Blinder [20]. We include it in the list of problems for completeness. We have the well known expressions

$$\begin{aligned} K^{(\mathrm{FP})}(x'', x'; \tau) &= \left(\frac{m}{2\pi\hbar\tau}\right)^{1/2} \exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right) \\ G^{(\mathrm{FP})}(x'', x'; s) &= \left(\frac{m}{2s}\right)^{1/2} \exp\left(-\frac{\sqrt{2ms}}{\hbar}|x''-x'|\right). \end{aligned} \tag{28}$$

We obtain immediately for the perturbed problem

$$\begin{aligned} G(x'', x'; s) = {} & \left(\frac{m}{2s}\right)^{1/2} \exp\left(-\frac{\sqrt{2ms}}{\hbar}|x''-x'|\right) \\ & + \frac{m\gamma}{2\hbar} \frac{\exp[-(\sqrt{2ms}/\hbar)(|x''-a| + |a-x'|)]}{\sqrt{s}\,(\sqrt{s} - (\gamma/\hbar)\sqrt{m/2})} \end{aligned} \tag{29}$$

For $\gamma > 0$ there is one bound state with wavefunction

$$\Psi(x) = \frac{\sqrt{m\gamma}}{\hbar} \exp\left(-\frac{m\gamma}{\hbar^2}|x-a|\right) \tag{30}$$

and energy

$$E=-\frac{m\gamma^2}{2\hbar^2}. \tag{31}$$

For $s>0$ we have a continuous spectrum of scattering states. Due to the simple form of $G(s)$, $K(\tau)$ can be explicitly stated and is given by

$$\begin{aligned}K(x'',x';\tau)=&\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right)\\&+\frac{m\gamma}{2\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{m\gamma^2\tau}{2\hbar^3}\right)\\&\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|+|x'-a|-\frac{\gamma\tau}{\hbar}\right)\right].\end{aligned} \tag{32}$$

Here the inverse Laplace transformation [21, p 247]

$$L^{-1}[p^{-1/2}(p^{1/2}+\beta)^{-1}\exp(-\alpha\sqrt{p})](t)=\exp(\alpha\beta+\beta t^2)\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}}+\beta\sqrt{t}\right)$$

has been used, $\operatorname{erfc}(x)=1-\operatorname{erf}(x)$ and $\operatorname{erf}(x)$ is the standard error function $\operatorname{erf}(x)=(2/\sqrt{\pi})\int_0^x\exp(-t^2)\,dt$. Without loss of generality let us set $a=0$. $K(\tau)$ can be rewritten as [19]

$$\begin{aligned}K(x'',x';\tau)=&\frac{m\gamma}{\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x'|+|x''|)+\frac{m\gamma^2\tau}{2\hbar^3}\right)\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\,\exp\left(-\frac{p^2\hbar\tau}{2m}\right)\\&\times\left(\sin px'\sin px''+\cos px'\cos px''-\frac{\exp[ip(|x'|+|x''|)]}{1+ip\hbar^2/m\gamma}\right)\end{aligned} \tag{33}$$

which gives odd scattering states which are not affected at all

$$\Psi_{\text{odd}}(x)=\frac{1}{\sqrt{2\pi}}\sin px \tag{34a}$$

and even scattering states which are affected by the δ potential

$$\Psi_{\text{even}}(x)=\frac{1}{\sqrt{2\pi}}\left(\cos px-\frac{e^{ip|x|}}{1+ip\hbar^2/m\gamma}\right). \tag{34b}$$

3.1.2. Free particle on the half-line. The propagator for the half-line ($\mathbb{R}^+$) is well known and has the form [2, 18, 22]

$$\begin{aligned}K^{(\mathbb{R}^+)}&(x'',x';\tau)\\&=K^{(\mathrm{FP})}(x'',x';\tau)-K^{(\mathrm{FP})}(x'',-x';\tau)\\&=\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\left[\exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right)-\exp\left(-\frac{m}{2\hbar\tau}(x''+x')^2\right)\right].\end{aligned} \tag{35}$$

I have chosen the boundary condition (see [22]) in such a way that the boundary at $x=0$ is totally reflecting and therefore the wavefunctions vanish at $x=0$. The Feynman kernel can be constructed by the method of images, or by the fact that a radial path integral can be written as a superposition (provided that the potential satisfies $V(r) = V(-r)$) of a sum of one-dimensional path integrals [18]. In this sense the half-line problem is a 'one-dimensional' radial problem with $l = 0$. For the Green function we obtain

$$G^{(\mathbb{R}^+)}(x'', x'; s) = \left(\frac{m}{2s}\right)^{1/2} \left[\exp\left(-\frac{\sqrt{2ms}}{\hbar}|x''-x'|\right) - \exp\left(-\frac{\sqrt{2ms}}{\hbar}|x''+x'|\right)\right]. \tag{36}$$

Therefore for the perturbed problem

$$\begin{aligned} G(x'', x'; s) &= \left(\frac{m}{2s}\right)^{1/2} [\exp(-|x''-x'|\sqrt{2ms}/\hbar) - \exp(-|x''+x'|\sqrt{2ms}/\hbar)] \\ &\quad + \frac{m\gamma}{2\hbar} \frac{[\exp(-|x''-a|\sqrt{2ms}/\hbar) - \exp[-|x''+a|\sqrt{2ms}/\hbar)]}{\sqrt{s}\,\{\sqrt{s} - (\gamma/\hbar)\sqrt{m/2}\,[1-\exp(-2a\sqrt{2ms}/\hbar)]\}} \\ &\quad \times [\exp(-|a-x'|\sqrt{2ms}/\hbar) - \exp(-|a+x'|\sqrt{2ms}/\hbar)]. \end{aligned} \tag{37}$$

We obtain a transcendental equation for the energy level

$$E = -\frac{m\gamma^2}{2\hbar^2}\left[1 - \exp\left(-\frac{2a}{\hbar}\sqrt{-2mE}\right)\right]^2. \tag{38}$$

This is the result of Van Siclen [23]. In particular, there is no effect for $a = 0$ because the wavefunctions must vanish for $x = 0$. For the limit $a \to \infty$

$$E \simeq -\frac{m\gamma^2}{2\hbar^2} \tag{39}$$

so the farther away the δ function the lesser the influence of the reflecting wall. A detailed numerical study of this transcendental equation can be found in [23], including a discussion of resonance states. Also it is found that in order that a bound state exists we must have $a\gamma > \hbar^2/2m$.

3.1.3. Infinite square well. As the next example I consider the infinite square well (ISW). Lapidus [24] calls it a 'one-dimensional hydrogen atom in an infinite square well'. Let us set

$$V(x) = \begin{cases} 0 & \text{for } |x| < b \\ \infty & \text{for } |x| \geqslant b. \end{cases} \tag{40}$$

The Feynman kernel for the infinite square well is given by [2, 25, 26]

$$\begin{aligned} K^{(\mathrm{ISW})}(x'', x'; \tau) &= \left(\frac{m}{2\pi\hbar\tau}\right)^{1/2} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{m}{2\hbar\tau}(x''-x'+2nb)^2\right) \right. \\ &\quad \left. - \exp\left(-\frac{m}{2\hbar\tau}(x''+x'+(2n+1)b)^2\right)\right] \\ &= \frac{1}{2b}\left[\Theta_3\left(\frac{x''-x'}{2b}, \frac{\mathrm{i}\pi\hbar\tau}{2mb^2}\right) - \Theta_3\left(\frac{x''+x'}{2b} + \frac{1}{2}, \frac{\mathrm{i}\pi\hbar\tau}{2mb^2}\right)\right] \\ &= \frac{2}{b}\sum_{n=1}^{\infty} \exp\left(-\hbar\tau\frac{\pi^2 n^2}{2mb^2}\right)\cos\left(\frac{\pi n}{b}x'\right)\cos\left(\frac{\pi n}{b}x''\right). \end{aligned} \tag{41}$$

Here Θ_3 denotes a Jacobi function [27, p 371]

$$\Theta_3(z,\tau)=\left(\frac{\mathrm{i}}{\tau}\right)^{1/2}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\mathrm{i}\pi}{\tau}(z+n)^2\right)=1+2\sum_{n=1}^{\infty}\cos(2\pi nz)\exp(\mathrm{i}\pi\tau n^2). \tag{42}$$

I obtain for the Green function

$$G^{(\mathrm{ISW})}(x'',x';s)=\left(\frac{m}{2s}\right)^{1/2}\frac{\cosh[\sqrt{2ms}\,(x''-x'-b)/\hbar]-\cosh[\sqrt{2ms}\,(x''+x')/\hbar]}{\sinh(\sqrt{2ms}\,b/h)} \tag{43}$$

where I have used the Laplace transformation [21, p 224]

$$\int_0^\infty \Theta_3\left(\frac{1}{2}+\frac{x}{2l},\frac{\mathrm{i}\pi\tau}{l^2}\right)\mathrm{e}^{-pt}\,\mathrm{d}t=\frac{1}{\sqrt{p}}\frac{\cosh(x\sqrt{p})}{\sinh(l\sqrt{p})}.$$

Therefore for the perturbed problem

$$\begin{aligned}G(x'',x';s)&=\left(\frac{m}{2s}\right)^{1/2}\frac{\cosh[\sqrt{2ms}\,(x''-x'-b)/\hbar]-\cosh[\sqrt{2ms}\,(x''+x')/\hbar]}{\sinh(\sqrt{2ms}\,b/h)}\\&\quad+\frac{m\gamma}{2s\hbar}\left[\cosh\left(\sqrt{2ms}\,\frac{x''-a-b}{\hbar}\right)-\cosh\left(\sqrt{2ms}\,\frac{x''+a}{\hbar}\right)\right]\\&\quad\times\frac{1}{\sinh(\sqrt{2ms}\,b/\hbar)}\left[\cosh\left(\sqrt{2ms}\,\frac{a-x'-b}{\hbar}\right)-\cosh\left(\sqrt{2ms}\,\frac{a+x'}{\hbar}\right)\right]\\&\quad\times\left\{\sinh\left(\sqrt{2ms}\,\frac{b}{\hbar}\right)-\frac{\gamma}{\hbar}\left(\frac{m}{2s}\right)^{1/2}\left[\cosh\left(\sqrt{2ms}\,\frac{b}{\hbar}\right)-\cosh\left(\sqrt{2ms}\,\frac{2a}{\hbar}\right)\right]\right\}^{-1}.\end{aligned} \tag{44}$$

In particular for $a=0$:

$$\begin{aligned}G(x'',x';s)&=\left(\frac{m}{2s}\right)^{1/2}\frac{\cosh[\sqrt{2ms}\,(x''-x'-b)/\hbar]-\cosh[\sqrt{2ms}\,(x''+x')/\hbar]}{\sinh(\sqrt{2ms}\,b/\hbar]}\\&\quad+\frac{m\gamma}{2s\hbar}\left\{\left[\cosh\left(\sqrt{2ms}\,\frac{x''-b}{\hbar}\right)-\cosh\left(\sqrt{2ms}\,\frac{x''}{\hbar}\right)\right]\right.\\&\quad\times\left.\left[\cosh\left(\sqrt{2ms}\,\frac{x'+b}{\hbar}\right)-\cosh\left(\sqrt{2ms}\,\frac{x'}{\hbar}\right)\right]\right\}\\&\quad\times\left\{2\sinh\left(\sqrt{2ms}\,\frac{b}{\hbar}\right)\sinh\left(\sqrt{\frac{ms}{2}}\,\frac{b}{\hbar}\right)\right.\\&\quad\times\left.\left[\cosh\left(\sqrt{\frac{ms}{2}}\,\frac{b}{\hbar}\right)-\frac{\gamma}{\hbar}\sqrt{\frac{m}{2s}}\sinh\left(\sqrt{\frac{ms}{2}}\,\frac{b}{\hbar}\right)\right]\right\}^{-1}.\end{aligned} \tag{45}$$

For $a\neq 0$ we have the transcendental equation for the energy levels E_n

$$\frac{\hbar}{\gamma}\left(\frac{2E_n}{m}\right)^{1/2}=\frac{\cos[(2a/\hbar)\sqrt{2mE_n}\,]-\cos[(b/\hbar)\sqrt{2mE_n}\,]}{\sin[(b/\hbar)\sqrt{2mE_n}\,]}. \tag{46}$$

For $a=0$ this yields

$$\frac{\hbar}{\gamma}\left(\frac{2E_n}{m}\right)^{1/2}=\tan\left[\frac{b}{\hbar}\left(\frac{mE_n}{2}\right)^{1/2}\right] \tag{47}$$

which is equivalent with ($E_n = \hbar^2 k_n^2/2m$):

$$\frac{\hbar^2 k_n}{m\gamma} = \begin{cases} \tan \dfrac{bk_n}{2} & (E_n > 0) \\ \tanh \dfrac{bk_n}{2} & (E_n < 0). \end{cases} \tag{48}$$

A numerical discussion of these features can be found in [24]. Note that for $a = 0$ only the even wavefunctions are perturbed; the odd wavefunctions remain unchanged. A similar effect appears if it happens that the δ function is located at a node of a particular wavefunction; this wavefunction is unaffected by the perturbation.

3.1.4. Harmonic oscillator. The perturbation of a harmonic oscillator (HO) by a δ function has been discussed, e.g., by Janke and Cheng [28], but they have mainly concentrated on the statistical properties of the system. We consider

$$W(x) = (m/2)\omega^2 x^2 - \gamma\delta(x-a). \tag{49}$$

The Feynman kernel and the Green function (e.g. [29, 30]) of the harmonic oscillator are given by ($x'' \geqslant x'$):

$$K^{(\mathrm{HO})}(x'', x'; \tau) = \left(\frac{m\omega}{2\pi\hbar \sinh \omega\tau}\right)^{1/2} \exp\left[-\frac{m\omega}{2\hbar}\left((x''^2 + x'^2)\coth \omega\tau - 2\frac{x'x''}{\sinh \omega\tau}\right)\right] \tag{50a}$$

$$G^{(\mathrm{HO})}(x'', x'; s) = -\frac{1}{2}\left(\frac{m}{\pi\hbar\omega}\right)^{1/2} \Gamma\left(\frac{1}{2} + \frac{s}{\hbar\omega}\right) D_{-1/2-s/\hbar\omega}\left(-\left(\frac{2m\omega}{\hbar}\right)^{1/2} x'\right) \times D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2} x''\right). \tag{50b}$$

Here $D_\nu(z)$ denote the parabolic cylinder functions defined by (e.g. [31, p 39])

$$D_\nu(z) = 2^{\nu/2} \exp(-z^2/4) \left[\frac{\Gamma(\frac{1}{2})}{\Gamma((1-\nu)/2)}\, {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \frac{z}{\sqrt{2}} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\nu/2)}\, {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right].$$

($G^{(\mathrm{HO})}$ can also be calculated by setting $\mathrm{e}^x = \sqrt{\pi x/2}\,[I_{1/2}(x) + I_{-1/2}(x)]$, applying the integral (56) below and using relations between the Whittaker and parabolic cylinder functions.) Therefore I obtain for the perturbed system (e.g. $x'' \geqslant a \geqslant x'$)

$$G(x'', x'; s) = -\frac{1}{2}\left(\frac{m}{\pi\hbar\omega}\right)^{1/2} \times \left[\Gamma\left(\frac{1}{2} + \frac{s}{\hbar\omega}\right) D_{-1/2-s/\hbar\omega}\left(-\left(\frac{2m\omega}{\hbar}\right)^{1/2} x'\right) D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2} x''\right)\right] \times \left[1 + \frac{\gamma}{2\hbar}\left(\frac{m}{\pi\hbar\omega}\right)^{1/2} \Gamma\left(\frac{1}{2} + \frac{s}{\hbar\omega}\right) \times D_{-1/2-s/\hbar\omega}\left(-\left(\frac{2m\omega}{\hbar}\right)^{1/2} a\right) D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2} a\right)\right]^{-1}. \tag{51}$$

Of course, the energy levels E_n are determined by the transcendental equation

$$0=1+\frac{\gamma}{2\hbar}\left(\frac{m}{\pi\hbar\omega}\right)^{1/2}\Gamma\left(\frac{1}{2}-\frac{E_n}{\hbar\omega}\right)D_{-1/2+E_n/\hbar\omega}\left(-\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right)$$
$$\times D_{-1/2+E_n/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right). \tag{52}$$

In particular, for $a=0$:

$$1+\frac{\gamma}{2\hbar}\left(\frac{m}{\pi\hbar\omega}\right)^{1/2}\Gamma\left(\frac{1}{2}-\frac{E_n}{\hbar\omega}\right)D^2_{-1/2+E_n/\hbar\omega}(0)=1+\frac{\gamma}{4\hbar}\left(\frac{m}{\hbar\omega}\right)^{1/2}\frac{\Gamma(\frac{1}{4}-E_n/2\hbar\omega)}{\Gamma(\frac{3}{4}-E_n/2\hbar\omega)}=0. \tag{53}$$

Here again only the even wavefunctions are affected; the odd ones remain unchanged.

3.1.5. Harmonic oscillator on the half-line. Let us consider the harmonic oscillator on the half-line (HO⁺):

$$V(x)=\begin{cases} m\omega^2x^2/2 & \text{for } x>0 \\ \infty & \text{for } x\leqslant 0. \end{cases} \tag{54}$$

The Feynman kernel can easily be constructed by the same methods of images as in subsection 3.1.2, yielding

$$K^{(\mathrm{HO}^+)}(x'',x';\tau)$$
$$=K^{(\mathrm{HO})}(x'',x';\tau)-K^{(\mathrm{HO})}(x'',-x';\tau)$$
$$=\frac{m\omega\sqrt{x'x''}}{\hbar\sinh\omega\tau}\exp\left(-\frac{m\omega}{2\hbar}(x''^2+x'^2)\coth\omega\tau\right)I_{1/2}\left(\frac{m\omega x'x''}{\hbar\sinh\omega\tau}\right). \tag{55}$$

To calculate the Green function we use the integral representation ([31, p 86; 32, p 729], $a_1>a_2$, $\mathrm{Re}(\frac{1}{2}+\mu-\nu)>0$):

$$\int_0^\infty \coth^{2\nu}\left(\frac{x}{2}\right)\exp\left(-\frac{a_1+a_2}{2}t\cosh x\right)I_{2\mu}(t\sqrt{a_1a_2}\sinh x)\,\mathrm{d}x$$
$$=\frac{\Gamma(\frac{1}{2}+\mu-\nu)}{t\sqrt{a_1a_2}\,\Gamma(1+2\mu)}W_{\nu,\mu}(a_1t)M_{\nu,\mu}(a_2t). \tag{56}$$

Here $W_{\nu,\mu}$ and $M_{\nu,\mu}$ denote Whittaker functions which are defined by [32, p. 1055]

$$W_{\nu,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)}M_{\nu,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\nu)}M_{\nu,-\mu}(z)\,.$$

and the $M_{\nu,\mu}(z)$ are given by $M_{\nu,\mu}(z)=z^{\mu+1/2}\,\mathrm{e}^{-z/2}{}_1F_1(\mu-\nu+\frac{1}{2};2\mu+1;z)$. We obtain $(x''\geqslant x')$

$$G^{(\mathrm{HO}^+)}(x'',x';s)$$
$$=\left(\frac{m}{2\pi\hbar\omega}\right)^{1/2}\Gamma\left(\frac{3}{4}+\frac{s}{2\hbar\omega}\right)2^{1/2+s/2\hbar\omega}D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}x''\right)$$
$$\times E^{(1)}_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}x'\right) \tag{57}$$

where [31, pp 40, 41]

$$E_\nu^{(1)}(z) = 2z\exp(-z^2/4)\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right) = 2\sqrt{2}\left(\frac{z^2}{2}\right)^{-1/4} M_{\nu/2+1/4,1/4}\left(\frac{z^2}{2}\right)$$

and $D_\nu(z) = 2^{\nu/2}(z^2/2)^{-1/4}W_{\nu/2+1/4,\pm 1/4}(z^2/2)$. Therefore I get for the perturbed problem (e.g. $x'' \geqslant a \geqslant x'$)

$$G(x'',x';s) = \left[2^{1/4+s/2\hbar\omega}\left(\frac{m}{2\pi\hbar\omega}\right)^{1/2}\Gamma\left(\frac{3}{4}+\frac{s}{2\hbar\omega}\right)D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}x''\right) \times E^{(1)}_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}x'\right)\right] \times\left[1-\frac{\gamma}{\hbar}\left(\frac{m}{2\pi\hbar\omega}\right)^{1/2}2^{1/4+s/2\hbar\omega}\Gamma\left(\frac{3}{4}+\frac{s}{2\hbar\omega}\right)D_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right) \times E^{(1)}_{-1/2-s/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right)\right]^{-1}. \tag{58}$$

The energy levels E_n are defined by the equation

$$1 = \frac{\gamma}{\hbar}\left(\frac{m}{2\pi\hbar\omega}\right)^{1/2}2^{1/4-E_n/2\hbar\omega}\Gamma\left(\frac{3}{4}-\frac{E_n}{2\hbar\omega}\right)D_{-1/2+E_n/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right) \times E^{(1)}_{-1/2+E_n/\hbar\omega}\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}a\right). \tag{59}$$

For $a=0$ there is no effect at all because $E_\nu^{(1)}(0)\equiv 0$; the wavefunctions vanish for $x=0$.

3.1.6. Morse potential and Liouville quantum mechanics. As the next example we consider the Morse potential (M)

$$V^{(\mathrm{M})}(x) = \frac{V_0^2\hbar^2}{2m}(\mathrm{e}^{2x}-2\alpha\,\mathrm{e}^{x}) \tag{60}$$

($V_0>0$ and $\alpha\in\mathbb{R}$ are constants). This potential has $N_{\max}<\alpha V_0-\frac{1}{2}$ bound states. The Green function can be calculated by the path integral formalism explicitly [33-36] and is ($x''\geqslant x'$)

$$G^{(\mathrm{M})}(x'',x';s) = \frac{m}{V_0\hbar}\frac{\Gamma(\frac{1}{2}+\sqrt{2ms}/\hbar-\alpha V_0)}{\Gamma(1+2\sqrt{2ms}/\hbar)\exp[(x'+x'')/2]} \times W_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{x''})M_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{x'}). \tag{61}$$

I obtain for the perturbed problem (e.g. $x''\geqslant a>x'$)

$$G(x'',x';s) = \frac{m}{V_0\hbar}\frac{\Gamma(\frac{1}{2}+\sqrt{2ms}/\hbar-\alpha V_0)}{\Gamma(1+2\sqrt{2ms}/\hbar)\exp[(x'+x'')/2]} \times[W_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{x''})M_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{x'})] \times\left[1-\frac{m\gamma}{\hbar^2}\frac{\Gamma(\frac{1}{2}+\sqrt{2ms}/\hbar-\alpha V_0)}{\Gamma(1+2\sqrt{2ms}/\hbar)\,\mathrm{e}^{a})} \times W_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{a})M_{\alpha V_0,\sqrt{2ms}/\hbar}(2V_0\,\mathrm{e}^{a})\right]^{-1}. \tag{62}$$

The energy levels E_n are determined by

$$\frac{V_0\hbar^2}{m\gamma}=\frac{\Gamma(\frac{1}{2}+\sqrt{-2mE_n}/\hbar-\alpha V_0)}{\Gamma(1+2\sqrt{-2mE_n}/\hbar)\,\mathrm{e}^a}\,W_{\alpha V_0,\sqrt{-2mE_n}/\hbar}(2V_0\,\mathrm{e}^a)M_{\alpha V_0,\sqrt{-2mE_n}/\hbar}(2V_0\,\mathrm{e}^a). \tag{63}$$

Liouville quantum mechanics [37-39] can be studied by setting $\alpha=0$ in $V^{(\mathrm{M})}$. With the help of $W_{0,\mu}(z)=\sqrt{2z/\pi}\,K_\mu(z/2)$ and $M_{0,\mu}(z)=2^{2\mu}\Gamma(1+\mu)\sqrt{z}\,I_\mu(z/2)$ [32, p 1062] we obtain (e.g. $x''\geqslant a\geqslant x'$)

$$G(x'',x';s)=\frac{2m}{\hbar}\frac{I_{\sqrt{2ms}/\hbar}(V_0\,\mathrm{e}^{x'})K_{\sqrt{2ms}/\hbar}(V_0\,\mathrm{e}^{x''})}{1-(2m\gamma/\hbar^2)I_{\sqrt{2ms}/\hbar}(V_0\,\mathrm{e}^a)K_{\sqrt{2ms}/\hbar}(V_0\,\mathrm{e}^a)}. \tag{64}$$

Again, the energy level is determined by ($\gamma>0$)

$$\frac{\hbar^2}{2m\gamma}=I_{\sqrt{-2mE}/\hbar}(V_0\,\mathrm{e}^a)K_{\sqrt{-2mE}/\hbar}(V_0\,\mathrm{e}^a). \tag{65}$$

For $\gamma<0$ there exists no bound state. We can study the limiting case $V_0\,\mathrm{e}^a\ll 1$, then with [40, p 119] $I_\nu(z)\simeq(z/2)^\nu/\Gamma(1+\nu)$ and $K_\nu(z)\simeq\frac{1}{2}(z/2)^{-\nu}\Gamma(\nu)(z\to 0)$:

$$\frac{\hbar^2}{2m\gamma}\simeq\frac{\Gamma(\sqrt{-2mE}/\hbar)}{\Gamma(1+\sqrt{-2mE}/\hbar)}=\frac{\hbar}{\sqrt{-2mE}}$$

and we recover in this limit the free particle case

$$E=-m\gamma^2/2\hbar^2.$$

(In the next order of approximation E can be determined by the cubic equation ($z\to 0$) $\nu^3-\nu^2(m\gamma/\hbar^2)-\nu+(m\gamma/\hbar^2)(1+z^2/2)=0$, where $\nu=\sqrt{-2mE}/\hbar$.) Since $I_0(z)K_0(z)\in(0,\infty)$ ($z>0$) there is always a solution $a_\gamma=a(\gamma,V_0)$ implicitly defined by ($\gamma>0$)

$$\frac{\hbar^2}{m\gamma}=I_0(V_0\,\mathrm{e}^a)K_0(V_0\,\mathrm{e}^a)$$

such that for all $a<a_\gamma$ there exists a bound state. The increasing potential well for $a\to\infty$ of Liouville quantum mechanics neutralizes the effect of the binding strength of the δ function and no bound state can exist for $a>a_\gamma$.

3.1.7. Reflectionless potential. The scattering properties of the potential

$$V(x)=-\frac{\hbar^2N(N+1)}{2m\cosh^2 x}\qquad N\in\mathbb{N} \tag{66}$$

have been studied by Crandell [41] and Crandell and Litt [42]. They called it 'reflectionless' because for an incoming continuous state $\propto\mathrm{e}^{\mathrm{i}px}$ ($p=$ momentum) there is *no* reflected contribution $\propto\mathrm{e}^{-\mathrm{i}px}$. This feature depends critically on the fact that N is a natural number. The path integral solution of this kind of potential can be obtained by the path integral for the modified Pöschl-Teller potential [7] and the bound and continuous states of the reflectionless (RL) potential can be written as (e.g. [7, 43, 44])

$$\begin{aligned}\Psi_n^{(N)}(x)&=\left((N-n)\frac{(2N+1-n)!}{n!}\right)^{1/2}P_N^{n-N}(\tanh x)\\ E_n&=-\frac{\hbar^2}{2m}(N-n)^2\end{aligned} \tag{67}$$

for bound states, and

$$\Psi_p^{(N)}(x)=\left(\frac{p}{2\sinh\pi p}\right)^{1/2}P_N^{\mathrm{i}p}(\tanh x)$$
$$E_p=\frac{\hbar^2p^2}{2m}\qquad p\in\mathbb{R} \tag{68}$$

for continuous states. Here $P_\nu^\mu(x)$ denote Legendre functions which are defined by ($|x|\leqslant 1$, [32, p 999]):

$$P_\nu^\mu(x)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2}{}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right).$$

Because $N\in\mathbb{N}$, the bound and continuous states are in fact polynomials in $\tanh x$ and $\mathrm{e}^{-x}/\cosh x$, respectively. For $N=1$ we obtain

$$\Psi_0^{(1)}(x)=\frac{1}{\sqrt{2}\cosh x}\qquad E_0=-\frac{\hbar^2}{2m}$$
$$\Psi_p^{(1)}(x)=\frac{\mathrm{e}^{\mathrm{i}px}}{\sqrt{2\pi}}\frac{\mathrm{i}p-\tanh x}{1+\mathrm{i}p}. \tag{69}$$

$N=2$ yields

$$\Psi_0^{(2)}(x)=\frac{\sqrt{3}}{2\cosh^2 x}\qquad E_0=-\frac{2\hbar^2}{m}$$
$$\Psi_1^{(2)}(x)=\sqrt{\frac{3}{2}}\frac{\sinh x}{\cosh^2 x}\qquad E_1=-\frac{\hbar^2}{2m} \tag{70}$$
$$\Psi_p^{(2)}(x)=\frac{\mathrm{e}^{\mathrm{i}px}}{\sqrt{2\pi}}\left(1-\frac{3}{1-\mathrm{i}p}\frac{\mathrm{e}^{-x}}{\cosh x}-\frac{1}{2(2-\mathrm{i}p)}\frac{\mathrm{e}^{-2x}}{\cosh^2 x}\right).$$

Let us first discuss the $N=1$ case because it is the simplest one. The Feynman kernel can be cast into closed form [41, 42] and is

$$\begin{aligned}
K^{(\mathrm{RL})}&(x'',x';\tau)\\
&=\Psi_0^{(1)}(x')\Psi_0^{(1)}(x'')\exp(-E_0\tau/\hbar)+\int_{-\infty}^{\infty}\mathrm{d}p\,\Psi_p^{(1)*}(x')\Psi_p^{(1)}(x'')\exp(-E_p\tau/\hbar)\\
&=\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right)\\
&\quad+\frac{\exp(\hbar\tau/2m)}{4\cosh x'\cosh x''}\left[\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}-(x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right.\\
&\quad\left.+\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}+(x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right].
\end{aligned} \tag{71}$$

Here use has been made of the integral representations [32, p 497] ($\operatorname{Re}\beta,\gamma>0$, $a>0$):

$$\begin{aligned}
\int_0^\infty &\frac{x\,\mathrm{d}x}{\gamma^2+x^2}\exp(-\beta x^2)\sin ax\\
&=-\frac{\pi}{4}\exp(\beta\gamma^2)\left[2\sinh a\gamma+\mathrm{e}^{-a\gamma}\operatorname{erf}\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-\mathrm{e}^{a\gamma}\operatorname{erf}\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]
\end{aligned}$$

$$\int_0^\infty \frac{dx}{\gamma^2+x^2}\exp(-\beta x^2)\cos ax$$

$$=\frac{\pi}{4\gamma}\exp(\beta\gamma^2)\left[2\cosh a\gamma - e^{-a\gamma}\operatorname{erf}\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-e^{a\gamma}\operatorname{erf}\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right].$$

For the Green function we obtain

$$G^{(RL)}(x'',x';s)$$

$$=\left(\frac{m}{2s}\right)^{1/2}\exp\left(-\frac{|x''-x'|}{\hbar}\sqrt{2ms}\right)+\frac{\hbar}{2\cosh x'\cosh x''}\frac{1}{s-\hbar^2/2m}$$

$$\times\left\{1-\left(1-\frac{\hbar}{\sqrt{2ms}}\right)\cosh\left[|x''-x'|\left(1+\frac{\sqrt{2ms}}{\hbar}\right)\right]\right\}. \tag{72}$$

As usual, this yields for the perturbed problem

$$G(x'',x';s)=G^{(RL)}(x',x';s)+\frac{\gamma}{\hbar}\frac{G^{(RL)}(x'',a;s)G^{(RL)}(a,x';s)}{1-(\gamma/\hbar)G^{(RL)}(a,a;s)} \tag{73}$$

and the energy levels E_n are determined by the equation

$$\frac{\hbar}{\gamma}=\left(-\frac{m}{2E_n}\right)^{1/2}-\frac{\hbar^2}{2\cosh^2 a\sqrt{-2mE_n}\,(E_n+\hbar^2/2m)}. \tag{74}$$

Here use has been made of the Laplace transformations (21, pp 176, 177]:

$$\int_0^\infty \exp[(a-p)t]\operatorname{erf}(\sqrt{at})\,dt=\left(\frac{a}{p}\right)^{1/2}\frac{1}{p-a}$$

$$\int_0^\infty \exp[(a-p)t]\operatorname{erfc}(\sqrt{at}+\tfrac{1}{2}\sqrt{\beta/t})=\frac{\exp(-\sqrt{a\beta}-\sqrt{p\beta})}{\sqrt{p}\,(\sqrt{p}+\sqrt{a})}.$$

Denoting $x=\sqrt{-E_n}$, this can be rewritten as a cubic equation in x:

$$x^3-\frac{\gamma}{\hbar}\left(\frac{m}{2}\right)^{1/2}x^2-\frac{\hbar^2}{2m}x+\frac{\gamma\hbar}{2\sqrt{2m}}\tanh^2 a=0. \tag{75}$$

An investigation of this equation shows that we have in general *three* bound states. For $a=0$, (75) reduces to

$$x\left(x^2-\frac{\gamma}{\hbar}\left(\frac{m}{2}\right)^{1/2}x-\frac{\hbar^2}{2m}\right)=0$$

which gives the two energy levels

$$E_1=-\left[\left(\frac{\hbar^2}{2m}+\frac{m\gamma^2}{8\hbar^2}\right)^{1/2}+\frac{\gamma}{2\hbar}\left(\frac{m}{2}\right)^{1/2}\right]^2$$

$$E_2=-\left[\left(\frac{\hbar^2}{2m}+\frac{m\gamma^2}{8\hbar^2}\right)^{1/2}-\frac{\gamma}{2\hbar}\left(\frac{m}{2}\right)^{1/2}\right]^2. \tag{76}$$

In the limit $a \to \infty$ (complete decoupling) we obtain

$$E_1 = -\frac{\hbar^2}{2m} \qquad E_2 = -\frac{m\gamma^2}{2\hbar^2}. \tag{77}$$

because the solutions of (75) are analytically known it is possible to state the Feynman kernel corresponding to $G(s)$ explicitly. However, this is rather tedious and lengthy and is postponed until the appendix. The (in general) three bound states can be easily stated by taking the residuum of (73) at $s = s_i$, which yields the wavefunctions

$$\Psi^{(j)}(x) = \frac{\gamma}{\hbar} \frac{2s_j(s_j - \hbar^2/2m)(\sqrt{s_3} - \sqrt{s_2})G^{(\mathrm{RL})}(x, a; s_j)}{\sqrt{s_2 s_3}\,(\sqrt{s_3} - \sqrt{s_2}) - \sqrt{s_1 s_3}\,(\sqrt{s_3} - \sqrt{s_1}) - \sqrt{s_1 s_2}\,(\sqrt{s_1} - \sqrt{s_2})} \tag{78}$$

corresponding to the energies $E = E_i = -s_i < 0$ $(i = 1, 2, 3)$.

Let us sketch the case for general $N \in \mathbb{N}$. According to [41] the Feynman kernel can be cast into a finite sum yielding

$$\begin{aligned} K^{(\mathrm{RL})}(x'', x'; \tau) &= \left(\frac{m}{2\pi\hbar\tau}\right)^{1/2} \exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right) + \frac{1}{2}\sum_{n=0}^{N-1} \Psi_n^{(N)}(x'')\Psi_n^{(N)*}(x') \\ &\quad \times \exp\left(\frac{\hbar\tau}{2m}(N-n)^2\right)\left[\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}(N-n) - (x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right. \\ &\quad \left. + \operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}(N-n) + (x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right]. \end{aligned} \tag{79}$$

The Green function is evaluated as

$$\begin{aligned} G^{(\mathrm{RL})}(x'', x'; s) &= \left(\frac{m}{2s}\right)^{1/2} \exp\left(-|x''-x'|\frac{\sqrt{2ms}}{\hbar}\right) + \frac{1}{2}\sum_{n=0}^{N-1} \frac{\Psi_n^{(N)}(x'')\Psi_n^{(N)*}(x')}{s - (\hbar^2/2m)(N-n)^2} \\ &\quad \times \left\{1 - \left(1 - \frac{\hbar(N-n)}{\sqrt{2ms}}\right)\cosh\left[|x''-x'|\left(N - n + \frac{\sqrt{2ms}}{\hbar}\right)\right]\right\}. \end{aligned} \tag{80}$$

The Green function of the perturbed problem is given by (73) with $G^{(\mathrm{RL})}$ of (80) instead of (72) and the energy levels E_n are determined by

$$\sqrt{-E_n} - \frac{m\gamma}{\hbar\sqrt{2m}} + \frac{\hbar\gamma}{\sqrt{2m}} \sum_{n=0}^{N-1} \frac{|\Psi_n^{(N)}(a)|^2}{(\hbar^2/2m)(N-n)^2 + E_n} = 0 \tag{81}$$

which is equivalent to finding the zeros of a polynomial of degree $(2N+1)$ in $\sqrt{-E_n}$, therefore in general giving rise to $(2N+1)$ energy levels.

3.1.8. Reflectionless potential on the half-line. For completeness let us study $V^{(\mathrm{RL})}$ on the half-line, i.e.

$$V^{(\mathrm{RL}^+)}(x) = \begin{cases} -\dfrac{\hbar^2 N(N+1)}{2m\cosh^2 m} & \text{for } x > 0 \\ \infty & \text{for } x \leqslant 0. \end{cases} \tag{82}$$

I state only the results for $N=1$. By the method of images we construct the Feynman kernel, yielding

$$K^{(\mathrm{RL}^+)}(x'', x'; \tau)$$

$$= \frac{1}{2}\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2} \exp\left(-\frac{m}{2\hbar\tau}(x'^2+x''^2)\right)\sinh\left(\frac{mx'x''}{\hbar\tau}\right)$$

$$+\frac{\exp(\hbar\tau/2m)}{4\cosh x'\cosh x''}\left[\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}-(x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right.$$

$$-\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}-(x''+x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)$$

$$+\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}+(x''-x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)$$

$$\left.-\operatorname{erf}\left(\left(\frac{\hbar\tau}{2m}\right)^{1/2}+(x''+x')\left(\frac{m}{2\hbar\tau}\right)^{1/2}\right)\right] \tag{83}$$

and the Green function:

$$G^{(\mathrm{RL}^+)}(x'', x'; s)$$

$$= \sqrt{\frac{m}{2s}}\left[\exp\left(-\frac{|x''-x'|}{\hbar}\sqrt{2ms}\right)-\exp\left(-\frac{|x''+x'|}{\hbar}\sqrt{2ms}\right)\right]$$

$$+\frac{\hbar(1-\hbar/\sqrt{2ms})}{\cosh x'\cosh x''}\frac{1}{s-\hbar^2/2m}\sinh\left(\frac{x'}{\hbar}\sqrt{2ms}\right)\sinh\left(\frac{x''}{\hbar}\sqrt{2ms}\right). \tag{84}$$

The Green function of the perturbed problem is given by

$$G(x'', x'; s) = G^{(\mathrm{RL}^+)}(x'', x'; s)+\frac{\gamma}{\hbar}\frac{G^{(\mathrm{RL}^+)}(x'', a; s)G^{(\mathrm{RL}^+)}(a, x'; s)}{1-(\gamma/\hbar)G^{(\mathrm{RL}^+)}(a, a; s)} \tag{85}$$

and the energy levels E_n are determined by the equation

$$\frac{\hbar}{\gamma}=\left(-\frac{m}{2E_n}\right)^{1/2}\left[1-\exp\left(-\frac{2a}{\hbar}\sqrt{-2mE_n}\right)\right]$$

$$+\frac{\hbar(1-\hbar/\sqrt{-2mE_n})}{\cosh^2 a(E_n+\hbar^2/2m)}\sinh^2\left(\frac{a}{\hbar}\sqrt{-2mE_n}\right). \tag{86}$$

Of course, for $a=0$, we have no effect at all.

3.1.9. Inverse distance potential. We consider the one-dimensional potential

$$V^{(q)}(x) = -q^2/|x| \qquad x\neq 0. \tag{87}$$

This 'Coulomb'-like potential separates, due to its strong singularity, the domains $x<0$ and $x>0$. Therefore it it is sufficient to consider only $V^{(q)}$ for $x>0$. Its Green function can be explicitly calculated and is $(x''\geqslant x')$

$$G^{(q)}(x'', x'; s) = \left(\frac{m}{2s}\right)^{1/2}\Gamma\left(1-\frac{q^2}{\hbar}\left(\frac{m}{2s}\right)^{1/2}\right)$$

$$\times W_{(q^2/\hbar)\sqrt{m/2s},1/2}\left(\frac{x''}{\hbar}\sqrt{8ms}\right)M_{(q^2/\hbar)\sqrt{m/2s},1/2}\left(\frac{x'}{\hbar}\sqrt{8ms}\right). \tag{88}$$

This result can be achieved by operator calculus [45] and path integration [46]. We obtain for the perturbed problem (e.g. $x''\geqslant a\geqslant x'$)

$$G(x'',x';s)=\left[\sqrt{\frac{m}{2s}}\Gamma\left(1-\frac{q^2}{\hbar}\sqrt{\frac{m}{2s}}\right)\right.$$
$$\times W_{(q^2/\hbar)\sqrt{m/2s},1/2}\left(\frac{x''}{\hbar}\sqrt{8ms}\right)M_{(q^2/\hbar)\sqrt{m/2s},1/2}\left.\left(\frac{x'}{\hbar}\sqrt{8ms}\right)\right]$$
$$\times\left[1-\frac{\gamma}{\hbar}\sqrt{\frac{m}{2s}}\Gamma\left(1-\frac{q^2}{\hbar}\sqrt{\frac{m}{2s}}\right)\right.$$
$$\times W_{(q^2/\hbar)\sqrt{m/2s},1/2}\left(\frac{a}{\hbar}\sqrt{8ms}\right)M_{(q^2/\hbar)\sqrt{m/2s},1/2}\left.\left(\frac{a}{\hbar}\sqrt{8ms}\right)\right]^{-1}. \tag{89}$$

The energy levels E_n are determined by the transcendental equation

$$\frac{\hbar}{\gamma}=\left(-\frac{m}{2E_n}\right)^{1/2}\Gamma\left(1-\frac{q^2}{\hbar}\left(-\frac{m}{2E_n}\right)^{1/2}\right)W_{(q^2/\hbar)\sqrt{-m/2E_n},1/2}\left(\frac{a}{\hbar}\sqrt{-8mE_n}\right)$$
$$\times M_{(q^2/\hbar)\sqrt{-m/2E_n},1/2}\left(\frac{a}{\hbar}\sqrt{-8mE_n}\right). \tag{90}$$

With the asymptotic expansions for $W_{\nu,\mu}(z)\simeq z^{\nu}\,e^{-z/2}$ and $M_{\nu,\mu}(z)\simeq z^{-\nu}\,e^{z/2}[\Gamma(\frac{1}{2}+\mu-\nu)\Gamma(1+2\mu)]^{-1}$ for $z\to\infty$ [31, pp 90, 91] the problem decouples and we obtain two spectra containing the energy levels

$$E_n=-\frac{mq^4}{2\hbar^2(n+1)^2}\quad(n=0,1,\ldots)\qquad E_\gamma=-\frac{m\gamma^2}{2\hbar^2}. \tag{91}$$

In the limit $a\to 0$ we find, with [31, p 28] $M_{\nu,\mu}(z)\simeq z/\Gamma(1+2\mu)$ and $M_{\nu,\mu}(z)\simeq 1/\Gamma(1-\nu)$ $(z\to 0)$,

$$1-\frac{\gamma}{\hbar}G^{(q)}(a,a;s)\simeq 1-\frac{2am\gamma}{\hbar^2}=1\qquad a\to 0$$

and the δ function has no effect, due to the vanishing of the wavefunctions at the origin.

3.1.10. Pöschl–Teller potential. The Pöschl–Teller potential is defined by

$$V^{(\mathrm{PT})}(x)=\frac{\hbar^2}{2m}\left(\frac{\kappa(\kappa-1)}{\sin^2 x}+\frac{\lambda(\lambda-1)}{\cos^2 x}\right)\qquad 0<x<\frac{\pi}{2}. \tag{92}$$

The path integral of this potential can be calculated with the help of the path integral on the SU(2) manifold [6, 7] and the Green function is given by

$$G^{(\mathrm{PT})}(x'',x';s)=\hbar\sum_{n=0}^{\infty}\frac{\Psi_n^{(\mathrm{PT})*}(x')\Psi_n^{(\mathrm{PT})}(x'')}{(\hbar^2/2m)(\kappa+\lambda+\frac{1}{2})^2+s} \tag{93}$$

with the wavefunctions

$$\Psi_n^{(\mathrm{PT})}(x)=\left(2(\kappa+\lambda+2n)\frac{n!\,\Gamma(\kappa+\lambda+n)}{\Gamma(\kappa+n+\frac{1}{2})\Gamma(\lambda+n+\frac{1}{2})}\right)^{1/2}$$
$$\times\sin^\kappa x\cos^\lambda x\,P_n^{(\kappa-1/2,\lambda-1/2)}(1-2\sin^2 x) \tag{94}$$

where $P_n^{(\alpha,\beta)}(x)$ denote Jacobi polynomials. Consequently we obtain for the perturbed problem

$$G(x'', x'; s) = G^{(\mathrm{PT})}(x'', x'; s) + \frac{\gamma}{\hbar} \frac{G^{(\mathrm{PT})}(x'', a; s) G^{(\mathrm{PT})}(a, x'; s)}{1 - (\gamma/\hbar) G^{(\mathrm{PT})}(a, a; s)} \tag{95}$$

and the energy levels E_n are determined by the equation

$$\frac{1}{\gamma} = \sum_{n=0}^{\infty} \frac{|\Psi_n^{(\mathrm{PT})}(a)|^2}{(\hbar^2/2m)(\kappa + \lambda + 2n)^2 - E_n}. \tag{96}$$

In contrast to the Green functions of the previous problems, this one cannot be stated in a closed form, but only as an infinite sum.

3.1.11. Modified Pöschl–Teller potential. The modified Pöschl–Teller potential is defined by

$$V^{(\mathrm{mPT})}(r) = \frac{\hbar^2}{2m} \left(\frac{\eta(\eta - 1)}{\sinh^2 r} - \frac{\nu(\nu - 1)}{\cosh^2 r} \right) \qquad r > 0. \tag{97}$$

This potential is a generalization of the reflectionless potential as discussed in subsection 3.1.7 and is, of course, far more complicated. Its path integral can be evaluated by means of the SU(1, 1) path integral [7] and the Green function is given by a sum of discrete and continuous states. We adopt the notation as in [43]. Let us define $2s = \eta(\eta - 1)$, $-2c = \nu(\nu - 1)$ and introduce the numbers k_1, k_2 which are defined in terms of s and c by $k_1 = \frac{1}{2}(1 \pm \sqrt{\frac{1}{4} - 2c})$, $k_2 = \frac{1}{2}(1 \pm \sqrt{\frac{1}{4} + 2s})$. The correct signs depend on the boundary conditions for $r \to 0$ and $r \to \infty$, respectively. In particular one gets for $s = 0$ an even and an odd wavefunction corresponding to $k_2 = \frac{1}{4}, \frac{3}{4}$. We obtain

$$\begin{aligned} G^{(\mathrm{mPT})}(r'', r'; s) = \hbar &\sum_{n=0}^{N_M} \frac{\Psi_n^{(k_1,k_2)*}(r') \Psi_n^{(k_1,k_2)}(r'')}{-(\hbar^2/2m)[(2k_1 - k_2 - n) - 1]^2 + s} \\ &+ \hbar \int_0^\infty \mathrm{d}p \frac{\Psi_p^{(k_1,k_2)*}(r') \Psi_p^{(k_1,k_2)}(r'')}{\hbar^2 p^2/2m + s} \end{aligned} \tag{98}$$

where the wavefunctions are given by

$$\begin{aligned} \Psi_n^{(k_1,k_2)}(r) &= \left(\frac{2n!(2k_1 - 1)\Gamma(2k_1 - n - 1)}{\Gamma(2k_2 + n)\Gamma(2k_1 - 2k_2 - n)} \right)^{1/2} (\sinh r)^{2k_2 - 1/2} (\cosh r)^{2n - 2k_1 + 3/2} \\ &\quad \times P_n^{[2k_1 - 1, 2(k_1 - k_2 - n) - 1]} \left(\frac{1 - \sinh^2 r}{\cosh^2 r} \right) \end{aligned} \tag{99}$$

$$\begin{aligned} \Psi_p^{(k_1,k_2)}(r) &= N_p^{(k_1,k_2)} (\cosh r)^{2k_1 - 1/2} (\sinh r)^{2k_2 - 1/2} \\ &\quad \times {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2 r) \\ N_p^{(k_1,k_2)} &= \frac{1}{\pi \Gamma(2k_2)} \left(\frac{p \sinh \pi p}{2} \right)^{1/2} [\Gamma(k_1 + k_2 - \kappa)\Gamma(-k_1 + k_2 + \kappa) \\ &\quad \times \Gamma(k_1 + k_2 + \kappa - 1)\Gamma(-k_1 + k_2 - \kappa + 1)]^{1/2}. \end{aligned} \tag{100}$$

Here $\kappa = \frac{1}{2}(1 + \mathrm{i}p)$ and $n = 0, 1, 2, \ldots, N_M$, $k_1 - k_2 - \frac{1}{2}$,

$$G(r'', r'; s) = G^{(\mathrm{mPT})}(r'', r'; s) + \frac{\gamma}{\hbar} \frac{G^{(\mathrm{mPT})}(r'', a; s) G^{(\mathrm{mPT})}(a, r'; s)}{1 - (\gamma/\hbar) G^{(\mathrm{mPT})}(a, a; s)} \tag{101}$$

and the energy levels E_n are determined by the expression

$$\frac{1}{\gamma}=\sum_{\nu=0}^{N_M}\frac{|\Psi_\nu^{(\mathrm{mPT})}(a)|^2\nu}{s-(\hbar^2/2m)[(2k_1-k_2-\nu)-1]^2-E_n}+\int_0^\infty \mathrm{d}p\frac{|\Psi_p^{(\mathrm{mPT})}(a)|^2}{(\hbar^2p^2/2m)-E_n} \tag{102}$$

again a quite complicated expression, where only very special values of the parameters allow significant simplification.

3.1.12. Free particle with δ function. As a final example for the one-dimensional case, where the Green function is known in closed form, we can study the effect of two δ functions, i.e.

$$W(x)=-\gamma\delta(x-a)-\tilde{\gamma}\delta(x-\tilde{a}) \tag{103}$$

and interpret the second δ function as a perturbation of the first (see also Zhdanov and Chikhachev [12] where such a system is understood as a particle in the field of two δ potentials which are flying apart). From subsection 3.1.1. we have the Feynman kernel and the Green function:

$$\begin{aligned}K^{(\delta)}&(x'',x';\tau)\\&=\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right)\\&\quad+\frac{m\gamma}{2\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{m\gamma^2\tau}{2\hbar^3}\right)\\&\quad\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|+|x'-a|-\frac{\gamma\tau}{\hbar}\right)\right]\end{aligned} \tag{33}$$

$$\begin{aligned}G^{(\delta)}&(x'',x';s)\\&=\left(\frac{m}{2s}\right)^{1/2}\exp\left[-\left(\frac{2ms}{\hbar^2}\right)^{1/2}|x''-x'|\right]\\&\quad+\frac{m\gamma}{2\hbar}\frac{\exp[-(\sqrt{2ms}/\hbar)(|x''-a|+|a-x'|)]}{\sqrt{s}\,(\sqrt{s}-(\gamma/\hbar)\sqrt{m/2})}.\end{aligned} \tag{29}$$

Thus we obtain the Green function for the double δ-function problem:

$$G(x'',x';s)=G^{(\delta)}(x'',x';s)+\frac{\tilde{\gamma}}{\hbar}\frac{G^{(\delta)}(x'',\tilde{a};s)G^{(\delta)}(\tilde{a},x';s)}{1-(\tilde{\gamma}/\hbar)G^{(\delta)}(\tilde{a},\tilde{a};s)}. \tag{104}$$

The energy levels E_n are determined by the equation

$$\frac{\hbar}{\tilde{\gamma}}=\left(-\frac{m}{2E_n}\right)^{1/2}+\frac{m\gamma}{2\hbar}\frac{\exp[-(2|\tilde{a}-a|/\hbar)\sqrt{-2mE_n}]}{\sqrt{-E_n}\,[\sqrt{-E_n}-(\gamma/\hbar)\sqrt{m/2}]}. \tag{105}$$

For either $|a|\to\infty$ or $|\tilde{a}|\to\infty$, respectively, the problem clearly decouples into two independent systems with energy levels

$$E=-\frac{m\gamma^2}{2\hbar^2}\qquad \tilde{E}=-\frac{m\tilde{\gamma}^2}{2\hbar^2}.$$

For $\tilde{a}=a$ we obtain, of course, the expected result

$$E=-\frac{m(\gamma+\tilde{\gamma})^2}{2\hbar^2}.$$

The whole procedure can be repeated for an arbitrary number of δ functions (including all the potential problems 3.1.2-3.1.9) but, of course, with increasing complexity. Let us state a recursion formula for this kind of problem. Let us consider

$$W(x)=V(x)-\sum_{j=1}^{N}\gamma_j\delta(x-a_j). \tag{106}$$

We set $G^{(0)}(s)\equiv G^{(V)}(s)$ and obtain for $j=1,\ldots,N$:

$$G^{(j)}(x'',x';s)=G^{(j-1)}(x'',x';s)+\frac{\gamma_j}{\hbar}\frac{G^{(j-1)}(x'',a_j;s)G^{(j-1)}(a_j,x';s)}{1-(\gamma_j/\hbar)G^{(j-1)}(a_j,a_j;s)}. \tag{107}$$

3.2. D-dimensional radial problems

3.2.1. Free particle. We first consider the simplest case $V(r)\equiv 0$, i.e. the free particle (FP). The Feynman kernel is given by

$$\begin{aligned}K_l^{(\mathrm{FP})}(r'',r';\tau)&\\ &=\frac{m}{\hbar\tau}(r'r'')^{(2-D)/2}\exp\left(-\frac{m}{2\hbar\tau}(r'^2+r''^2)\right)I_{l+(D-2)/2}\left(\frac{mr'r''}{\hbar\tau}\right)\\ &=(r'r'')^{(2-D)/2}\int_0^\infty p\,\mathrm{d}p\exp\left(-\frac{\hbar p^2\tau}{2m}\right)J_{l+(D-2)/2}(pr')J_{l+(D-2)/2}(pr'').\end{aligned} \tag{108}$$

The Green function is, with the help of the integral representation [32, p 719],

$$\int_0^\infty\frac{\mathrm{d}x}{x}\exp(-a/x-bx)J_\nu(x)=2J_\nu[\sqrt{2a(\sqrt{b^2+c^2}-b)}]K_\nu[\sqrt{2a(\sqrt{b^2+c^2}+b)}]$$

given by $(r''\geqslant r')$

$$G_l^{(\mathrm{FP})}(r'',r';s)=\frac{2m}{\hbar}(r'r'')^{(2-D)/2}I_{l+(D-2)/2}\left(\frac{r''}{\hbar}\sqrt{2ms}\right)K_{l+(D-2)/2}\left(\frac{r'}{\hbar}\sqrt{2ms}\right). \tag{109}$$

This gives for the perturbed problem (e.g. $r''\geqslant a\geqslant r'$)

$$\begin{aligned}G_l(r'',r';s)&=\frac{2m}{\hbar}(r'r'')^{(2-D)/2}\\ &\times\frac{I_{l+(D-2)/2}((r'/\hbar)\sqrt{2ms})K_{l+(D-2)/2}((r''/\hbar)\sqrt{2ms})}{1-(2ma\gamma/\hbar^2)I_{l+(D-2)/2}((a/\hbar)\sqrt{2ms})K_{l+(D-2)/2}((a/\hbar)\sqrt{2ms})}.\end{aligned} \tag{110}$$

The energy levels E_n are determined by

$$\frac{\hbar^2}{2ma\gamma}=I_{l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{-2mE_n}\right)K_{l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{-2mE_n}\right). \tag{111}$$

From the behaviour of $I_\nu(z)$ and $K_\nu(z)$ for $\nu\to\infty$, [40, p 122]:

$$I_\nu(\nu z)\simeq\frac{1}{\sqrt{2\pi\nu}}\frac{\mathrm{e}^{\nu\eta}}{(1+z^2)^{1/4}}\qquad K_\nu(\nu z)\simeq\left(\frac{\pi}{2\nu}\right)^{1/2}\frac{\mathrm{e}^{-\nu\eta}}{(1+z^2)^{1/4}}$$

where $\eta=(1+z^2)+\ln[z/(1+\sqrt{1+z^2})]$, we have for large l

$$\frac{\hbar^2}{2ma\gamma}\simeq\left[\left(l+\frac{D-2}{2}\right)^2+\frac{2ma^2|E_n|}{\hbar^2}\right]^{-1/2}$$

and only a finite number of values of l can contribute to bound states.

3.2.2. Two-dimensional particle confined in a sector. We can study a two-dimensional example by studying the potential

$$V(\phi)=\begin{cases}0 & \text{if } 0<\phi<\alpha \\ \infty & \text{elsewhere.}\end{cases} \tag{112}$$

Here, of course, two-dimensional polar coordinates $x=r\cos\phi$, $y=r\sin\phi$, $(r>0, \phi\in[0,2\pi])$ are used. The Feynman kernel has been first calculated by Crandel [47] and later on by path integration by Chetouani *et al* [48]. It has the form

$$K^{(\alpha)}(r'',\phi'',r',\phi';\tau)=\sum_{l=1}^{\infty}\sin\left(\frac{l\pi}{\alpha}\phi''\right)\sin\left(\frac{l\pi}{\alpha}\phi'\right)K_l^{(\alpha)}(r'',r';\tau) \tag{113}$$

where the radial kernel is given by

$$K_l^{(\alpha)}(r'',r';\tau)=\frac{2m}{\hbar\tau\alpha}\exp\left(-\frac{m}{2\hbar\tau}(r'^2+r''^2)\right)I_{l\pi/\alpha}\left(\frac{mr'r''}{\hbar\tau}\right). \tag{114}$$

The special example of $\alpha=2\pi$ has also been considered by Schulman [49]. For the Green function we obtain $(r''\geqslant r')$

$$G_l^{(\alpha)}(r'',r';\tau)=\frac{4m}{\hbar\alpha}I_{l\pi/\alpha}\left(\frac{r'}{\hbar}\sqrt{2ms}\right)K_{l\pi/\alpha}\left(\frac{r''}{\hbar}\sqrt{2ms}\right) \tag{115}$$

which yields, in the usual way, for the perturbed problem

$$G_l(r'',r';s)=G_l^{(\alpha)}(r'',r';s)+\frac{\gamma}{\hbar}\frac{G_l^{(\alpha)}(r'',a;s)G_l^{(\alpha)}(a,r';s)}{1-(\gamma/\hbar)G_l^{(\alpha)}(a,a;s)}. \tag{116}$$

The energy levels E_n are determined by the equation

$$\frac{\hbar}{\gamma}=\frac{4m}{\hbar\alpha}I_{l\pi/\alpha}\left(\frac{a}{\hbar}\sqrt{-2mE_n}\right)K_{l\pi/\alpha}\left(\frac{a}{\hbar}\sqrt{-2mE_n}\right). \tag{117}$$

From the behaviour of the modified Bessel functions I_ν and K_ν for $\nu\to\infty$ (see the previous section), we see that (117) yields approximately

$$\frac{2m\gamma}{\hbar^2 l\pi}\simeq\left(1+\frac{2ma^2\alpha^2|E_n|}{l^2\pi^2\hbar^2}\right)^{1/2}$$

so that only a finite number of levels can exist and that roughly $2m\gamma/\hbar^2\pi>1$ is required in order that even one single level can exist.

3.2.3. Radial harmonic oscillator. Next we consider the radial harmonic oscillator (RHO) in D dimensions which is given by

$$V(r)=\frac{m}{2}\omega^2r^2. \tag{118}$$

The Feynman kernel and the Green function are given by [3, 50] $(r''\geqslant r'$, cf (56)):

$$\begin{aligned}K_l^{(\mathrm{RHO})}(r'',r';\tau)\\ &=\frac{m\omega(r'r'')^{(2-D)/2}}{\hbar\sinh\omega\tau}\exp\left(-\frac{m\omega}{2\hbar}(r'^2+r''^2)\coth\omega\tau\right)\\ &\quad\times I_{l+(D-2)/2}\left(\frac{m\omega r'r''}{\hbar\sinh\omega\tau}\right)\end{aligned} \tag{119}$$

$$G_l^{(\mathrm{RHO})}(r'', r'; s)$$
$$= \frac{\Gamma[\frac{1}{2}(l+D/2+s/\hbar\omega)]}{\omega(r'r'')^{D/2}\Gamma(l+D/2)} W_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} r''^2\right)$$
$$\times M_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} r'^2\right). \tag{120}$$

Therefore we obtain for the perturbed problem (e.g. $r'' \geqslant a \geqslant r'$)

$$G_l(r'', r'; s) = \frac{\Gamma[\frac{1}{2}(l+D/2+s/\hbar\omega)]}{\Gamma(l+D/2)\omega(r'r'')^{D/2}}$$
$$\times W_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} r''^2\right) M_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} r'^2\right)$$
$$\times\left[1 - \frac{\gamma}{a\hbar\omega} \frac{\Gamma[\frac{1}{2}(l+D/2+s/\hbar\omega)]}{\Gamma(l+D/2)} W_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} a^2\right)\right.$$
$$\left.\times M_{-s/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} a^2\right)\right]^{-1}. \tag{121}$$

The energy eigenvalues E_n are given by

$$\frac{a\hbar\omega}{\gamma} = \frac{\Gamma[\frac{1}{2}(l+D/2-E_n/\hbar\omega)]}{\Gamma(l+D/2)} W_{E_n/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} a^2\right)$$
$$\times M_{E_n/2\hbar\omega,(1/2)(l+(D-2)/2)}\left(\frac{m\omega}{\hbar} a^2\right). \tag{122}$$

3.2.4. Inverse distance (Coulomb) potential. The inverse distance potential in D dimensions is described by

$$V(r) = -q^2/r \tag{123}$$

which, for $D = 3$ is just the Coulomb potential. For general $l \in \mathbb{R}$ this potential is also known as the Kratzer potential. The radial Green function for this problem can be evaluated by path integrals in closed form and is [8, 46, 51-53] ($r'' \geqslant r'$):

$$G_l^{(q)}(r'', r'; s) = (r'r'')^{(1-D)/2}\left(\frac{m}{2s}\right)^{1/2} \frac{\Gamma(l+(D-1)/2-(q^2/\hbar)\sqrt{m/2s})}{(2l+D-1)!}$$
$$\times W_{(q^2/\hbar)\sqrt{m/2s},l+(D-2)/2}\left(\frac{r''}{\hbar}\sqrt{8ms}\right)$$
$$\times M_{(q^2/\hbar)\sqrt{m/2s},l+(D-2)/2}\left(\frac{r'}{\hbar}\sqrt{8ms}\right). \tag{124}$$

For the perturbed problem we therefore obtain (e.g. $r \geqslant a \geqslant r'$)

$$G_l(r'', r'; s) = \frac{\Gamma(l+(D-1)/2-(q^2/\hbar)\sqrt{m/2s})}{(2l+D-1)!(r'r'')^{(D-1)/2}}\left(\frac{m}{2s}\right)^{1/2}$$
$$\times\left[W_{(q^2/h)\sqrt{m/2s},l+(D-2)/2}\left(\frac{r''}{\hbar}\sqrt{8ms}\right) M_{(q^2/\hbar)\sqrt{m/2s},l+(D-2)/2}\left(\frac{r'}{\hbar}\sqrt{8ms}\right)\right]$$
$$\times\left[1 - \frac{\gamma}{\hbar}\sqrt{\frac{m}{2s}} \frac{\Gamma(l+(D-1)/2-(q^2/\hbar)\sqrt{m/2s})}{(2l+D-1)!}\right.$$
$$\left.\times W_{(q^2/\hbar)\sqrt{m/2s},l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{8ms}\right) M_{(q^2/\hbar)\sqrt{m/2s},l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{8ms}\right)\right]^{-1}. \tag{125}$$

The energy levels E_n are determined by

$$\frac{\hbar}{\gamma}=\left(-\frac{m}{2E_n}\right)^{1/2}\frac{\Gamma(l+(D-1)/2-(q^2/\hbar)\sqrt{-m/2E_n})}{(2l+D-1)!}$$
$$\times W_{(q^2/\hbar)\sqrt{-m/2E_n},l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{-8mE_n}\right)$$
$$\times M_{(q^2/\hbar)\sqrt{-m/2E_n},l+(D-2)/2}\left(\frac{a}{\hbar}\sqrt{-8mE_n}\right). \qquad (126)$$

Of course, the method of subsection 3.1.12 of constructing the Green function for a one-dimensional potential problem with an arbitrary number of δ functions can also be applied for the D-dimensional problems, corresponding to a shelf structure of δ perturbations. For example, let

$$W(r)=V(r)-\gamma\delta(r-a)-\tilde{\gamma}\delta(r-\tilde{a}) \qquad (127)$$

then we obtain

$$G_l(r'',r';s)=G_l^{(V-\delta)}(r'',r';s)+\frac{\tilde{\gamma}}{\hbar}\frac{G_l^{(V-\delta)}(r'',a;s)G^{(V-\delta)}(a,r';s)}{1-(\tilde{\gamma}/\hbar)a^{D-1}G_l^{(V-\delta)}(a,a;s)}. \qquad (128)$$

4. Summary

In this paper I have discussed several examples (one- and D-dimensional) of exact summation of a perturbation series for potential problems with a δ-function perturbation. The expansion in a perturbation series uses the path integration technique. The main result is a (in general transcendental) equation for the perturbed energy levels E_n which in the one-dimensional case has the form

$$\frac{\hbar}{\gamma}=\mathrm{i}G^{(V)}(a,a;E_n) \qquad (129)$$

and in the D-dimensional radial case has the form

$$\frac{\hbar}{a^{D-1}\gamma}=\mathrm{i}G_l^{(V)}(a,a;E_n) \qquad (130)$$

where $G^{(V)}$ and $G_l^{(V)}(E)$ denote the Green functions of the unperturbed one- and D-dimensional problems, respectively, which are known, of course, by path integration. These two simple equations determine the energy levels in a unique way. However, the Green functions are not explicit in the sense that wavefunctions and energy levels can be expressed explicitly in terms of known functions (of, e.g., the level parameter n). But this is not surprising. More important is the fact that it is nevertheless possible to state in closed form a Green function whose poles give information about the spectrum.

Of course, every usual solvable quantum mechanical problem where the Feynman kernel or the Green function is known (at least even semiclassically) can be perturbed by δ functions and discussed by our technique. The selection presented in this paper was motivated by the fact that in the examples the Green function $G(E)$ is known in closed form or as a spectral expansion into discrete and continuous states, respectively.

In particular for the (modified) Pöschl–Teller potential there is a great variety of problems which can be treated by the technique of spacetime transformations [4, 8, 54–56], e.g. the potential $V(x)=V_0/(1+e^{-x})$ [57, 58], the Rosen–Morse potential $V^{(RM)}(x)=A\tanh x-B/\cosh^2 x$ [58–61], the Manning–Rosen potential $V^{(MR)}(r)=-A\coth r+B/\sinh^2 r$ [58], the Hulthén potential $V^{(H)}(r)=-V_0\,e^{-r/a}/(1-e^{-r/a})$ [62] or the Kepler problems in a space of constant negative or positive curvature [59, 63]. In all these cases the solution of subsection 3.1 can be applied; however, the various energy spectra must be taken into account. In this sense the examples here can be seen as basic examples; all others can be derived from them.

The case of a point δ-function perturbation in D-dimensional space, i.e. set $a=0$ in sections 2.2 and 3.2, must be treated with a modified δ perturbation [9, 10]. In this sense the result of [64] cannot be seen as correct; in the formalism the entire Green function must be taken, not only (the discrete) part of it. But this will be discussed elsewhere.

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Appendix. Propagator for the δ function perturbed reflectionless potential

In order to calculate the Feynman kernel $K(\tau)$ of the perturbed problem we split up the Green function according to

$$G(x'',x';s)=G^{\mathrm{I}}(x'',x';s)+G^{\mathrm{II}}(x'',x';s) \tag{A.1}$$

where $G^{\mathrm{I}}(s)$ denotes the free-particle part. Clearly

$$K^{\mathrm{I}}(x'',x';\tau)=\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\exp\left(-\frac{m}{2\hbar\tau}(x''-x')^2\right). \tag{A.2}$$

Let us denote the (analytically known) solutions of (75) by $x_1=\sqrt{s_1}$, $x_2=\sqrt{s_2}$ and $x_3=\sqrt{s_3}$. This gives for the second part of the perturbed Green function

$$\begin{aligned}
G^{\mathrm{II}}(x'',x';s)&=\frac{\gamma}{\hbar}\frac{G^{(RL)}(x'',a;s)G^{(RL)}(a,x';s)}{1-(\gamma/\hbar)G^{(RL)}(a,a;s)}\\
&=\frac{\gamma}{\hbar}\frac{\sqrt{s}\,(s-\hbar^2/2m)}{(\sqrt{s}-\sqrt{s_1})(\sqrt{s}-\sqrt{s_2})(\sqrt{s}-\sqrt{s_3})}G^{(RL)}(x'',a;s)G^{(RL)}(a,x';s)\\
&=\frac{\gamma}{\hbar}\frac{\sqrt{s_3}-\sqrt{s_2}}{\sqrt{s_2s_3}\,(\sqrt{s_3}-\sqrt{s_2})-\sqrt{s_1s_3}\,(\sqrt{s_3}-\sqrt{s_1})-\sqrt{s_1s_2}\,(\sqrt{s_1}-\sqrt{s_2})}\\
&\quad\times\left(G^{(1)}(x'',x';s)-\frac{\sqrt{s_3}-\sqrt{s_1}}{\sqrt{s_3}-\sqrt{s_2}}G^{(2)}(x'',x';s)\right.\\
&\quad\left.-\frac{\sqrt{s_1}-\sqrt{s_2}}{\sqrt{s_3}-\sqrt{s_2}}G^{(3)}(x'',x';s)\right)
\end{aligned} \tag{A.3}$$

where the Green functions $G^{(i)}(s)$ are given by $(i=1,2,3)$

$$G^{(i)}(x'',x';s)=\frac{\sqrt{s}}{\sqrt{s}-\sqrt{s_i}}\left(s-\frac{\hbar^2}{2m}\right)G^{(\mathrm{RL})}(x'',a;s)G^{(\mathrm{RL})}(a,x';s)$$
$$=\sum_{k=1}^{3}G^{(k,i)}(x'',x';s) \tag{A.4}$$

with $(i=1,2,3)$

$$G^{(1,i)}(x'',x';s)$$
$$=\frac{m}{2}\frac{\sqrt{s}}{\sqrt{s}-\sqrt{s_i}}\exp[-\sqrt{2ms}\,(|x''-a|+|x'-a)|)/\hbar]$$
$$-\frac{\hbar^2}{4\sqrt{s}\,(\sqrt{s}-\sqrt{s_i})}\exp[-\sqrt{2ms}\,(|x''-a|+|x'-a|)/\hbar] \tag{A.5}$$

$$G^{(2,i)}(x'',x';s)$$
$$=\left(\frac{m}{2}\right)^{1/2}\frac{\hbar}{2\cosh a\cosh x'}\,\frac{\exp(-\sqrt{2ms}\,|x''-a|/\hbar)}{\sqrt{s}-\sqrt{s_i}}$$
$$\times\left\{1-\left(1-\frac{\hbar}{\sqrt{2ms}}\right)\cosh\left[|x''-a|\left(1+\frac{\sqrt{2ms}}{\hbar}\right)\right]\right\}$$
$$+\left(\frac{m}{2}\right)^{1/2}\frac{\hbar}{2\cosh a\cosh x''}\,\frac{\exp(-\sqrt{2ms}\,|x'-a|/\hbar)}{\sqrt{s}-\sqrt{s_i}}$$
$$\times\left\{1-\left(1-\frac{\hbar}{\sqrt{2ms}}\right)\cosh\left[|x''-a|\left(1+\frac{\sqrt{2ms}}{\hbar}\right)\right]\right\} \tag{A.6}$$

$$G^{(3,i)}(x'',x';s)$$
$$=\frac{\hbar^2}{4\cosh^2 a\cosh x'\cosh x''}\,\frac{\sqrt{s}}{\sqrt{s}-\sqrt{s_i}}\,\frac{1}{s-(\hbar^2/2m)}$$
$$\times\left\{1-\left(1-\frac{\hbar}{\sqrt{2ms}}\right)\cosh\left[|x'-a|\left(1+\frac{\sqrt{2ms}}{\hbar}\right)\right]\right\}$$
$$\times\left\{1-\left(1-\frac{\hbar}{\sqrt{2ms}}\right)\cosh\left[|x''-a|\left(1+\frac{\sqrt{2ms}}{\hbar}\right)\right]\right\}. \tag{A.7}$$

Therefore

$$K^{\mathrm{II}}(x'',x';\tau)=\sum_{k,i=1}^{3}K^{(k,i)}(x'',x';\tau). \tag{A.8}$$

We make use of the inverse Laplace transformations [21, pp 246, 247]:

$$L^{-1}\left(\frac{\mathrm{e}^{-a\sqrt{p}}}{\sqrt{p}+b}\right)(t)=\frac{\mathrm{e}^{-a^2/4t}}{\sqrt{\pi t}}-b\,\mathrm{e}^{ab+b^2t}\,\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right)$$

$$L^{-1}\left(\frac{\sqrt{p}\,\mathrm{e}^{-a\sqrt{p}}}{\sqrt{p}+b}\right)(t)=\frac{1}{\sqrt{\pi}\,t}\left(\frac{a}{2\sqrt{t}}-b\sqrt{t}\right)\mathrm{e}^{-a^2/4t}+b^2\,\mathrm{e}^{ab+b^2t}\,\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right).$$

For $K^{(1,i)}(\tau)$ we obtain

$$K^{(1,i)}(x'',x';\tau)$$

$$=\frac{m}{2\hbar\tau}\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\left(|x''-a|+|a-x'|+\tau\left(\frac{2s_i}{m}\right)^{1/2}\right)$$

$$\times\exp\left(-\frac{m}{2\hbar\tau}(|x''-a|+|a-x'|)^2\right)$$

$$+\left(\frac{ms_i}{2\hbar^2}-\frac{1}{4}\right)\exp\left(-\frac{\sqrt{2ms_i}}{\hbar}(|x''-a|+|a-x'|)+\frac{s_i\tau}{\hbar}\right)$$

$$\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|+|a-x'|)-\tau\left(\frac{2s_i}{\hbar}\right)^{1/2}\right)\right]. \tag{A.9}$$

Similarly for $K^{(2,i)}(\tau)$:

$$K^{(2,i)}(x'',x';\tau)$$

$$=\frac{1}{2\cosh a\cosh x'}\left\{\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\left[\exp\left(-\frac{m}{2\hbar\tau}(x''-a)^2\right)\right.\right.$$

$$-\frac{1}{2}\exp\left(|x'-a|-\frac{m}{2\hbar\tau}(|x''-a|+|x'-a|)^2\right)$$

$$\left.-\frac{1}{2}\exp\left(|x'-a|-\frac{m}{2\hbar\tau}(|x''-a|-|x'-a|)^2\right)\right]$$

$$+\left(\frac{ms_i}{2\hbar^2}\right)^{1/2}\exp\left(|x''-a|\frac{\sqrt{2ms_i}}{\hbar}+\frac{s_i\tau}{\hbar}\right)$$

$$\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|-\tau\left(\frac{2s_i}{\hbar}\right)^{1/2}\right)\right]$$

$$+\frac{1}{4}\left(1+\frac{\sqrt{2ms_i}}{\hbar\tau}\right)\exp\left(|x'-a|-(|x''-a|+|a-x'|)\frac{\sqrt{2ms_i}}{\hbar}+\frac{s_i\tau}{\hbar}\right)$$

$$\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|+|a-x'|-\tau\left(\frac{2s_i}{\hbar}\right)^{1/2}\right)\right]$$

$$+\frac{1}{4}\left(1+\frac{\sqrt{2ms_i}}{\hbar\tau}\right)\exp\left(-|x'-a|-(|x''-a|-|a-x'|)\frac{\sqrt{2ms_i}}{\hbar}+\frac{s_i\tau}{\hbar}\right)$$

$$\left.\times\operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|-|a-x'|-\tau\left(\frac{2s_i}{\hbar}\right)^{1/2}\right)\right]\right\}+(x''\leftrightarrow x'). \tag{A.10}$$

In order to calculate $K^{(3,i)}(\tau)$ we split $G^{(3,i)}(s)$ into three contributions according to

$$G^{(3,i)}(x'',x';s)$$

$$=\frac{1}{s_i-(\hbar^2/2m)}\left[\frac{G^{(3,i,1)}(x'',x';s)}{\sqrt{s}-\sqrt{s_i}}+\frac{1}{2}\left(1+\frac{\sqrt{2ms_i}}{\hbar}\right)\frac{G^{(3,i,2)}(x'',x';s)}{\sqrt{s}-\hbar/\sqrt{2m}}\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{\sqrt{2ms_i}}{\hbar}\right)\frac{G^{(3,i,3)}(x'',x';s)}{\sqrt{s}+\hbar/\sqrt{2m}}\right]. \tag{A.11}$$

Thus

$$K^{(3,i)}(x'', x'; \tau) = \sum_{j=1}^{3} K^{(3,i,j)}(x'', x'; \tau). \tag{A.12}$$

This yields for each j, where $p_j = s_i, \pm\hbar^2/2m$ $(j = 1, 2, 3)$

$$\begin{aligned}
&K^{(3,i,j)}(x'', x'; \tau)\\
&\quad= \frac{1}{4\cosh^2 a \cosh x' \cosh x''} \Bigg[\!\!\Bigg[\exp(p_j\tau/\hbar)\,\mathrm{erfc}\left[-\left(\frac{p_j\tau}{\hbar}\right)^{1/2}\right]\\
&\quad+\frac{1}{2}\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\left\{\left(\frac{\hbar}{m}-\frac{|x''-a|}{\tau}+\left(\frac{2p_j}{m}\right)^{1/2}\right)\exp\left(|x''-a|-\frac{m}{2\hbar\tau}(x''-a)^2\right)\right.\\
&\quad+\left(\frac{\hbar}{m}+\frac{|x''-a|}{\tau}+\left(\frac{2p_j}{m}\right)^{1/2}\right)\exp\left(-|x''-a|-\frac{m}{2\hbar\tau}(x''-a)^2\right)\\
&\quad+\left(\frac{\hbar}{m}-\frac{|x'-a|}{\tau}+\left(\frac{2p_j}{m}\right)^{1/2}\right)\exp\left(|x'-a|-\frac{m}{2\hbar\tau}(x'-a)^2\right)\\
&\quad\left.+\left(\frac{\hbar}{m}+\frac{|x'-a|}{\tau}+\left(\frac{2p_j}{m}\right)^{1/2}\right)\exp\left(-|x'-a|-\frac{m}{2\hbar\tau}(x'-a)^2\right)\right\}\\
&\quad+\frac{1}{2}\left(\left(\frac{p_j}{2m}\right)^{1/2}-\frac{p_j}{\hbar}\right)\left\{\exp\left[|x''-a|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]\right.\\
&\quad\times\mathrm{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]\\
&\quad+\exp\left[-|x''-a|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]\\
&\quad\times\mathrm{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(-|x''-a|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]\\
&\quad+\exp\left[|x'-a|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]\\
&\quad\times\mathrm{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x'-a|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]\\
&\quad+\exp\left[-|x'-a|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]\\
&\quad\left.\times\mathrm{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(-|x'-a|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]\right\}\\
&\quad+\left(\frac{m}{2\pi\hbar\tau}\right)^{1/2}\left\{\left[\frac{1}{4}\left(\frac{2p_j}{m}\right)^{1/2}+\frac{|x''-a|+|a-x'|}{4\tau}-\frac{\hbar}{m}\right]\right.\\
&\quad\times\exp\left(|x''-a|+|a-x'|-\frac{m}{2\hbar\tau}(|x''-a|+|a-x'|)^2\right)\\
&\quad+\left[\frac{1}{4}\left(\frac{2p_j}{m}\right)^{1/2}+\frac{|x''-a|-|a-x'|}{4\tau}-\frac{\hbar}{m}\right]
\end{aligned}$$

$$\times \exp\left(|x''-a|-|a-x'|-\frac{m}{2\hbar\tau}(|x''-a|-|a-x'|)^2\right)$$
$$+\left[\frac{1}{4}\left(\frac{2p_j}{m}\right)^{1/2}-\frac{|x''-a|+|a-x'|}{4\tau}-\frac{\hbar}{m}\right]$$
$$\times \exp\left(-|x''-a|-|a-x'|-\frac{m}{2\hbar\tau}(|x''-a|+|a-x'|)^2\right)$$
$$+\left[\frac{1}{4}\left(\frac{2p_j}{m}\right)^{1/2}-\frac{|x''-a|-|a-x'|}{4\tau}-\frac{\hbar}{m}\right]$$
$$\times \exp\left(-|x''-a|+|a-x'|-\frac{m}{2\hbar\tau}(|x''-a|-|a-x'|)^2\right)\Bigg\}$$
$$+\left[\frac{\hbar}{2m}+\frac{p_j}{4\hbar}+\left(\frac{p_j}{2m}\right)^{1/2}\right]\Bigg\{\exp\left[|x''-a|+|a-x'|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|+|a-x'|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]$$
$$+\exp\left[|x''-a|-|a-x'|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(|x''-a|-|a-x'|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]$$
$$+\exp\left[-|x''-a|-|a-x'|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(-|x''-a|-|a-x'|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]$$
$$+\exp\left[-|x''-a|+|a-x'|\left(1+\frac{\sqrt{2mp_j}}{\hbar}\right)+\frac{p_j\tau}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\left(\frac{m}{2\hbar\tau}\right)^{1/2}\left(-|x''-a|+|a-x'|-\tau\left(\frac{2p_j}{m}\right)^{1/2}\right)\right]\Bigg\}\Bigg]\!\!\Bigg]. \qquad \text{(A.13)}$$

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